# Cosmology

B.F. Roukema

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- ■     verbal averaging: can we do better?

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- scalar (GR) averaging: statistically homogeneous spatial slices

# verbal averaging

- w:Cosmological principle

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- [w:Cosmological principle](#)

- practical meaning:

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- <u>w:Cosmological principle</u>

- practical meaning:

    1. assume homogeneity and isotropy

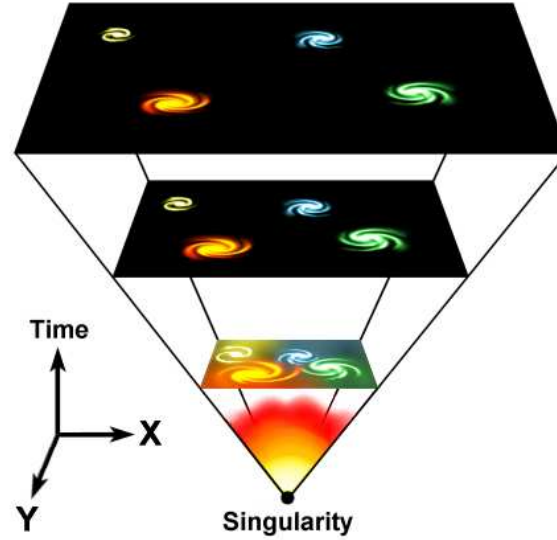# verbal averaging

■ <span style="color:green">w:Cosmological principle</span>

■ practical meaning:

1. assume homogeneity and isotropy

2. find the differential 4-pseudo-manifolds $(M, \mathbf{g})$ that solve $\mathbf{G} = 8\pi\mathbf{T}$, where
   - $M = $ a topological manifold
   - $g = $ a (pseudo-Riemannian) metric

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3. assume that $(M, \mathbf{g})$ remains unchanged if we add density perturbations to an early time slice
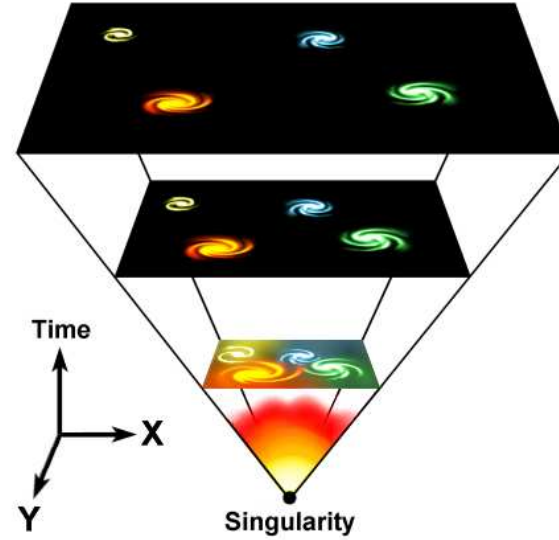
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- ■ [w:Comoving coordinates](w:Comoving coordinates)

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Time

X

Y

Singularity
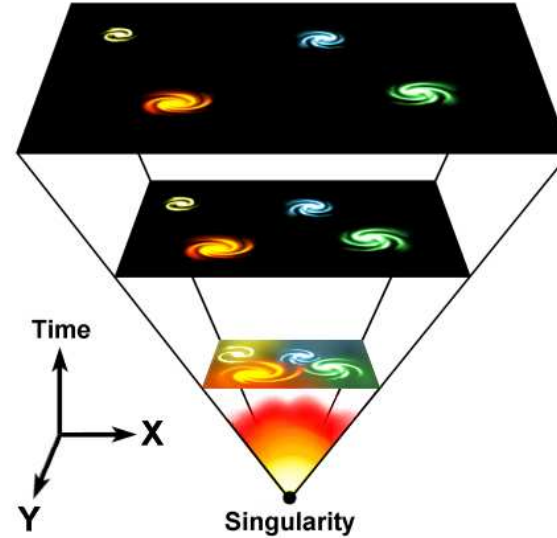
■ w:Comoving coordinates

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$$\Delta x(t) = a(t)\Delta r$$
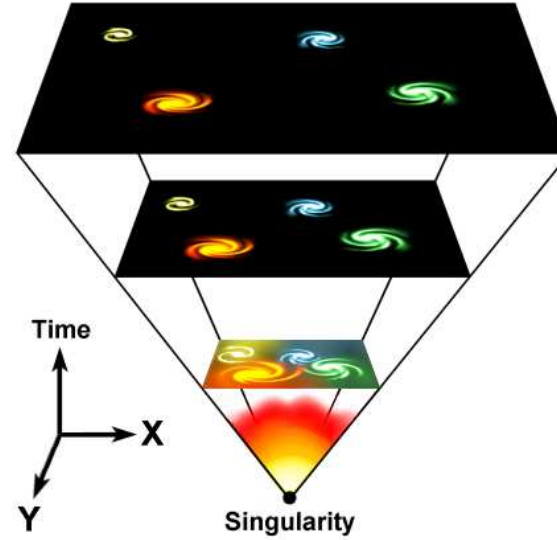
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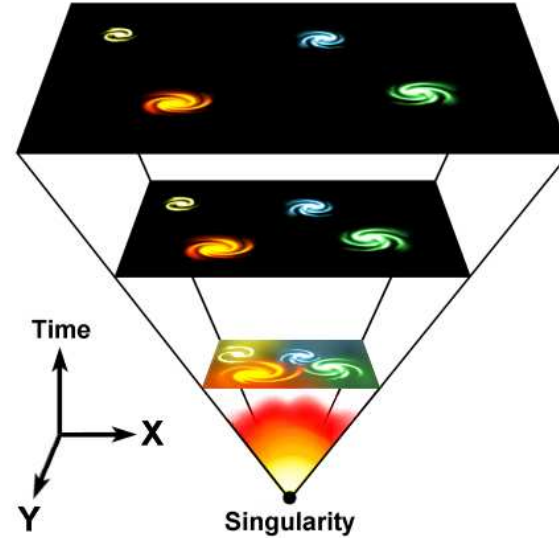- spherical coordinates for spatial slice

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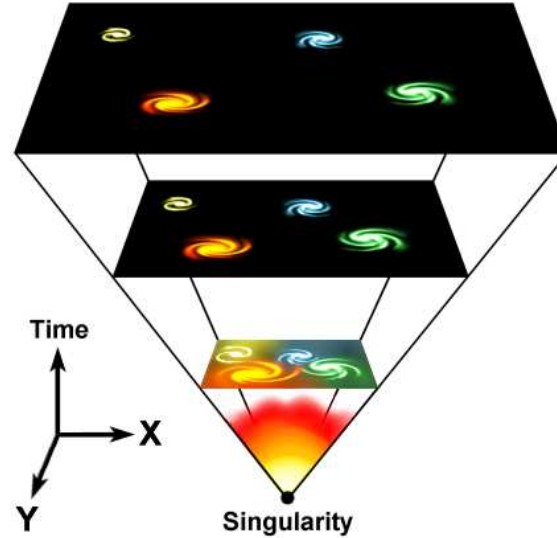
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- universe is static in comoving coordinates $(r,\theta,\phi)$

# FLRW metric

- [w:Friedmann–Lemaître–Robertson–Walker metric](#)

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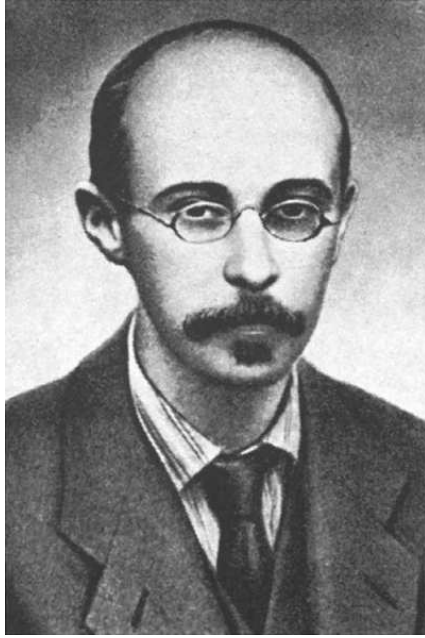


- [w:](#)    [w:](#)

[w:Howard Percy Robertson](#)

[w:Arthur Geoffrey Walker](#)

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■ but $\int \mathrm{d}u \neq$ proper time; *more:*   arXiv:0707.2106

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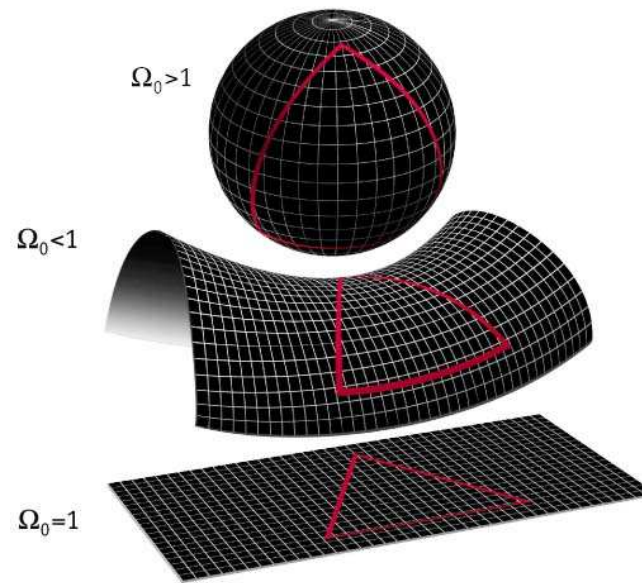


$$\text{where} \quad r_\perp := \begin{cases} R_\mathrm{C}\sinh\frac{r}{R_\mathrm{C}} & k < 0 \\ r & k = 0 \\ R_\mathrm{C}\sin\frac{r}{R_\mathrm{C}} & k > 0 \end{cases}$$

for a comoving radius of curvature $R_\mathrm{C}$ and curvature of sign $k$

# curvature

- on a spatial slice (fixed value of $t$):

$$\sqrt{x^2 + y^2}$$

y
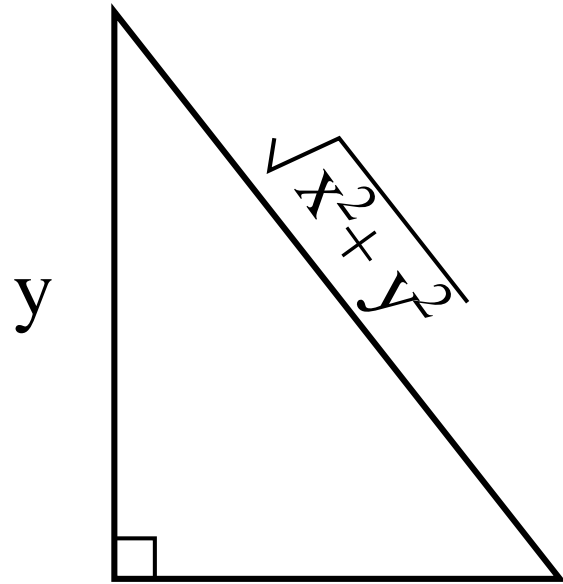
x

$$k = 0$$

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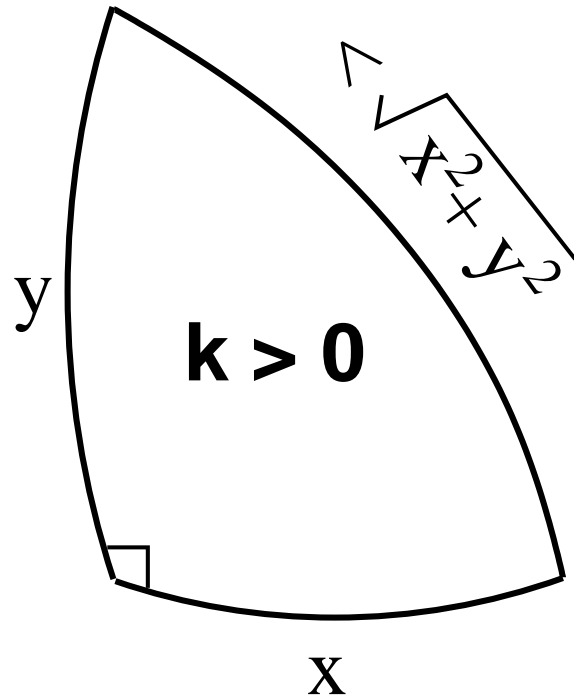
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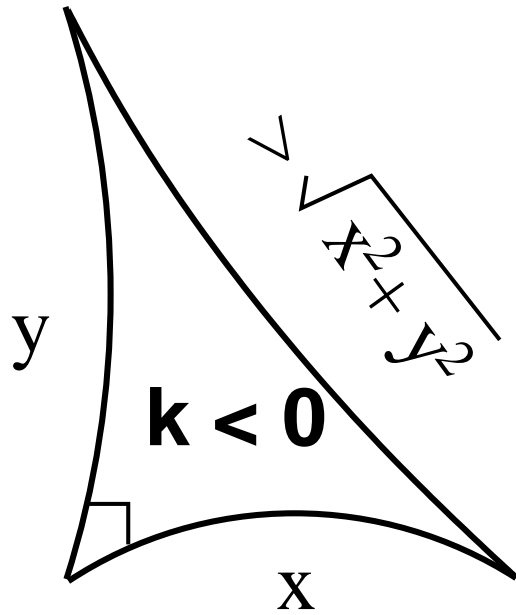


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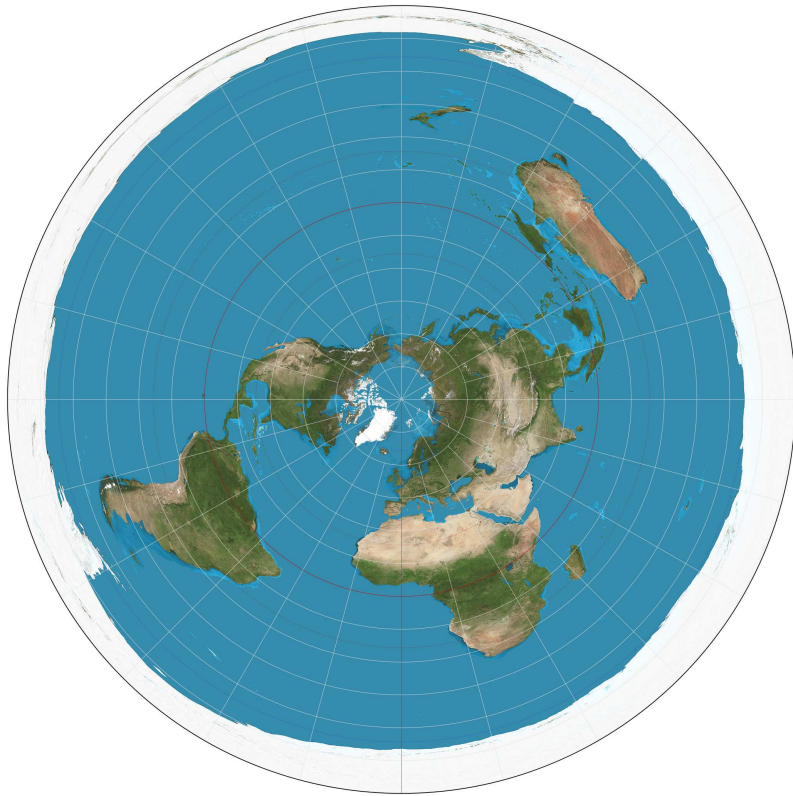


w: _____ (al-Biruni, c. 1000 CE)

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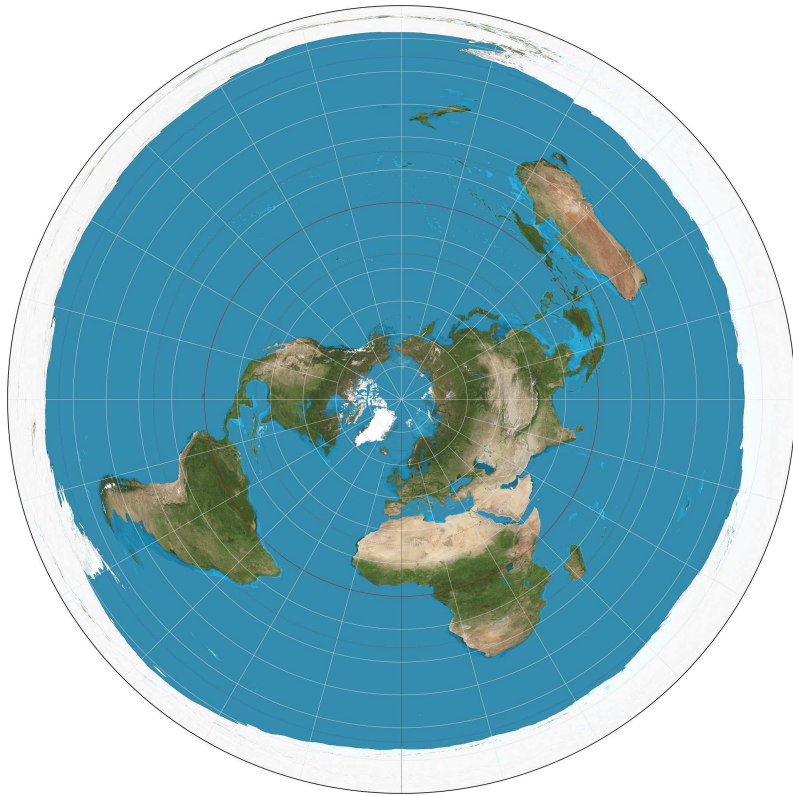


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■ intuition switch: $S^2$ easier vs $S^3$ more physical

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W:

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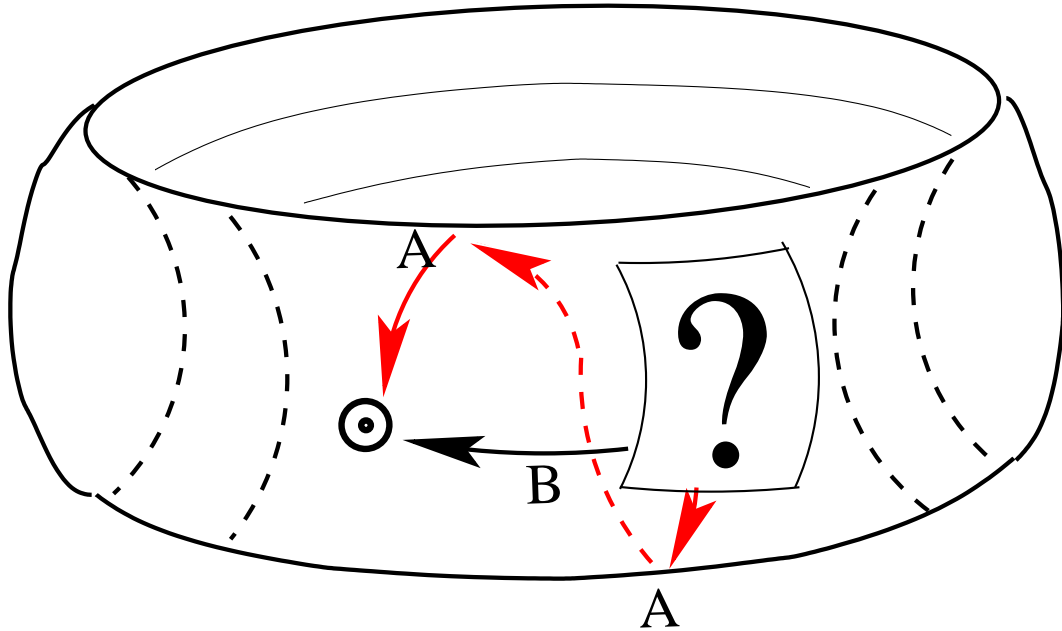
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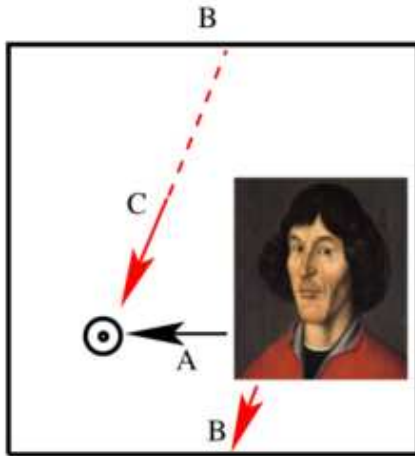
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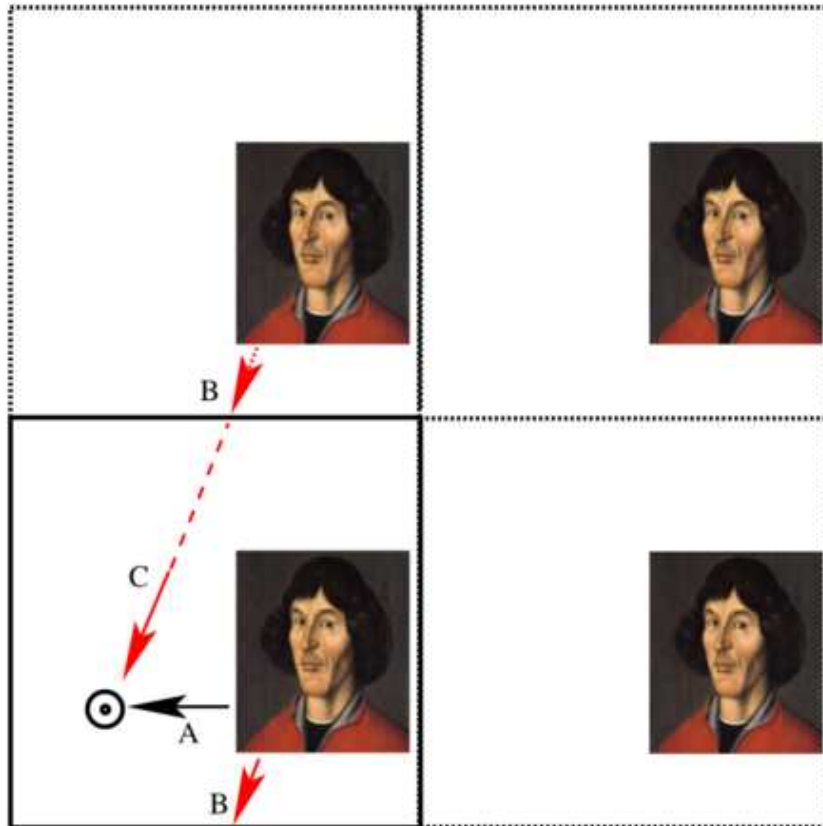
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■ convention: $t_{\mathrm{b}} := 0$, giving $t \to 0^+ \Rightarrow a(t) \to 0^+$

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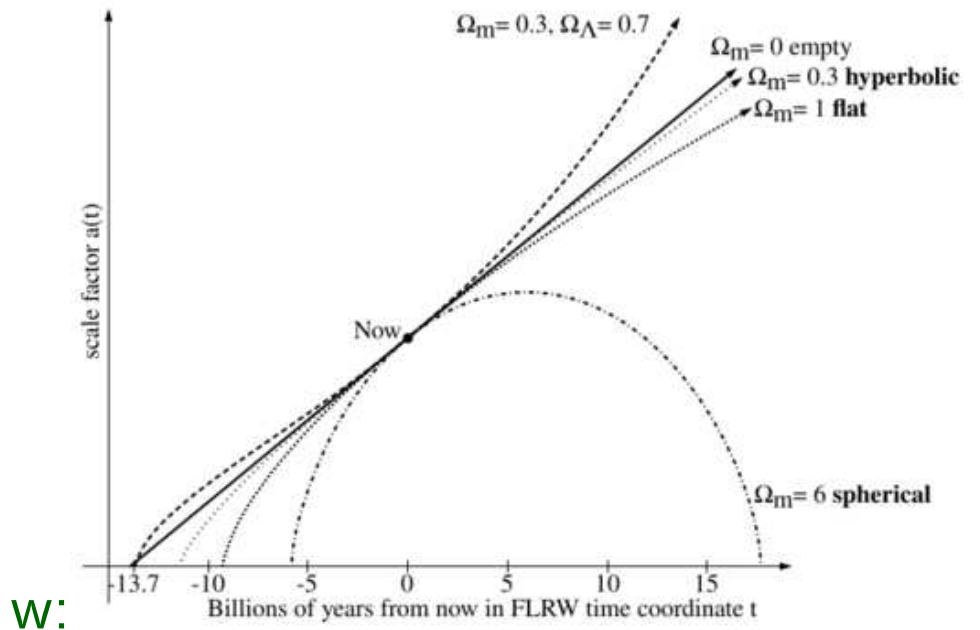
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$\Leftrightarrow$

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(Defn: $a_0 := 1$)

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$$1 + z = \frac{1}{a_{\mathrm{em}}}$$

(Defn of redshift $z$)

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(light-cone convention: $a$ often means $a_{\mathrm{em}}$)

- radiation density: $E = h\nu \Rightarrow \rho_{\mathrm{r}} \propto a^{-4} = (1 + z)^4$

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- $\Rightarrow$ black body + primordial nucleosynthesis

# CMB discovery: McKellar 1941

- $T \approx 2.3$ K — Andrew McKellar (1941; ADS:1941PDAO....7..251M) from observations by Walter S. Adams (1941; ADS:1941ApJ....93...11A)

- Penzias & Wilson rediscovery (1965 + Nobel prize)

# Black body: COBE ($\sim 1992$)

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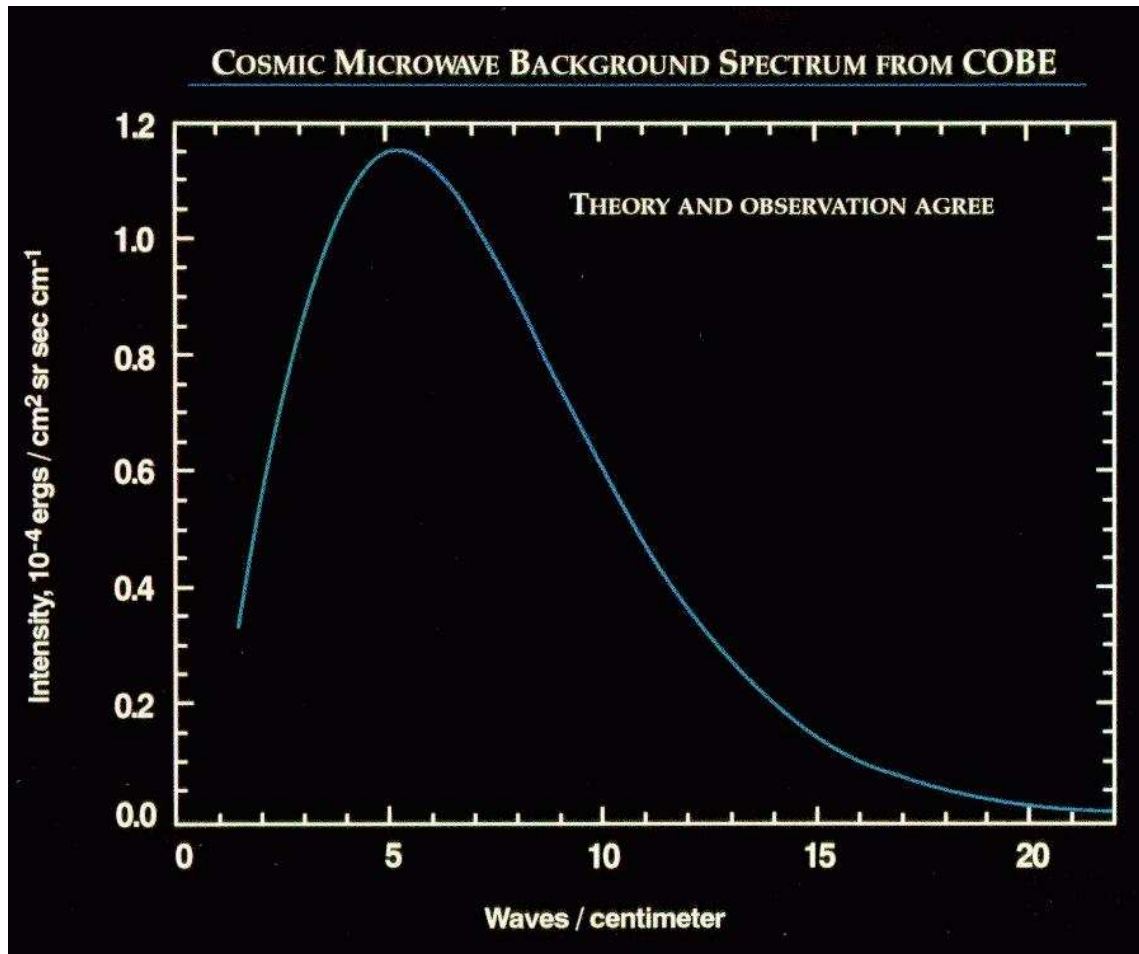
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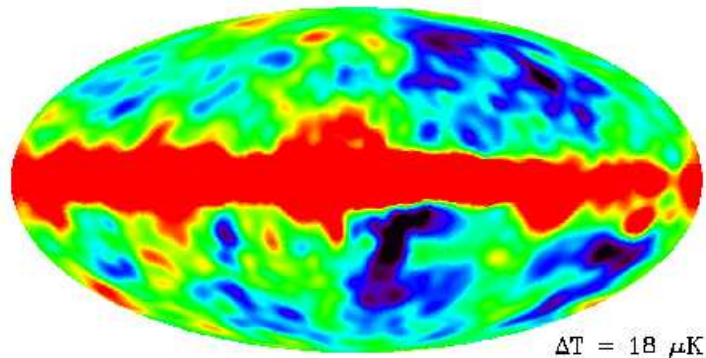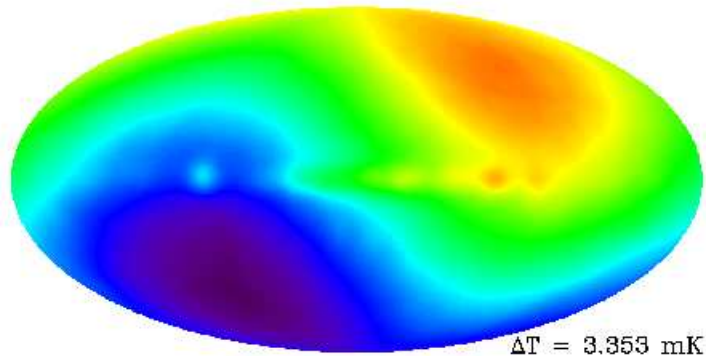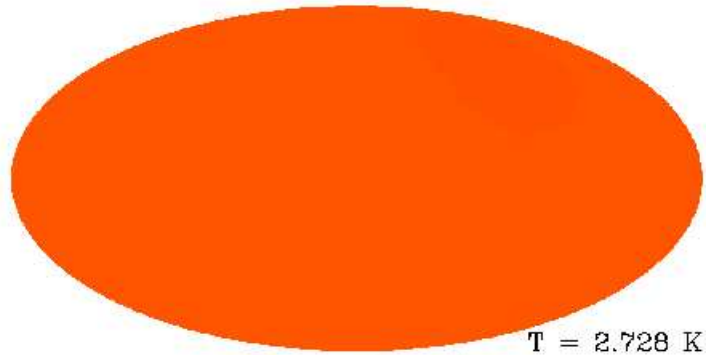
# BBN: Big bang nucleosynthesis

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main reactions of varying probability:

$$p + n \quad \rightarrow {}^2H + \gamma$$
$$p + {}^2H \quad \rightarrow {}^3He + \gamma$$
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\end{aligned}
$$
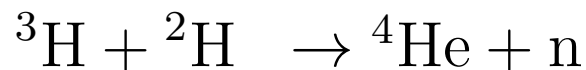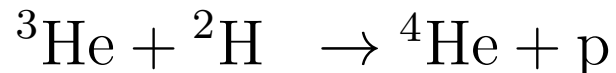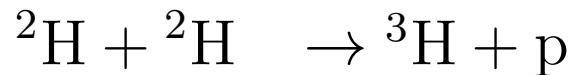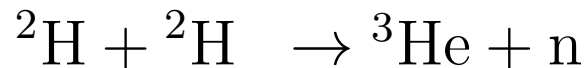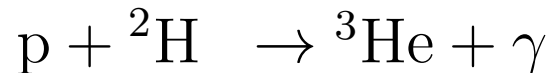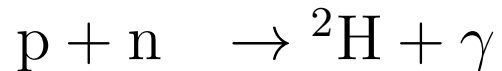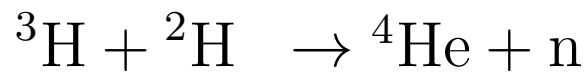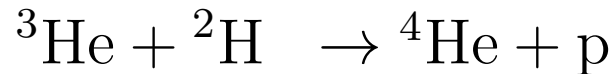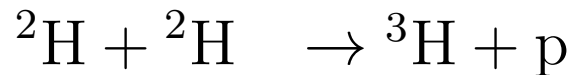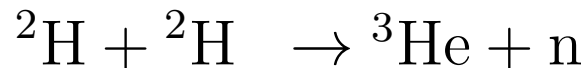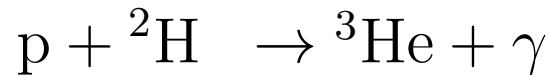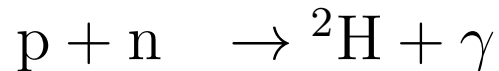
w:Big Bang nucleosynthesis

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■ convenient conversion: 1 km/s $\approx 1.04$ kpc/Gyr $\approx 1$ kpc/Gyr

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc

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# FLRW: $\rho_{\mathrm{crit}}$

■ Friedmann Eqn: $\boxed{1 = \Omega_{\mathrm{tot}} + \Omega_k}$

Defn: $\boxed{\rho_{\mathrm{crit}} := \frac{3H^2}{8\pi G}}$ critical density

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# FLRW curvature constant

- metric in

  - ◆ <u>azimuthal equidistant</u> coords: $R_C$

# FLRW curvature constant

- metric in

    - <u>azimuthal equidistant</u> coords: $R_C$

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# FLRW curvature constant

■ metric in

    ◆ <u>azimuthal equidistant</u> coords: $R_C$

    ◆ <u>orthographic</u> coords: $k$

■ orthographic: $1 - kr_\perp^2 = 0$ coord singularity at equator

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- $\Rightarrow kR_{\mathrm{C}}^2 = 1$

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- <u>Defn</u>: $\Omega_k := -\dfrac{c^2 k}{a^2 H^2} \;\Rightarrow\; k = -\dfrac{\Omega_{k0} H_0^2}{c^2}$

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- orthographic: $1 - k r_\perp^{\,2} = 0$ coord singularity at equator

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- $\Omega_{\mathrm{tot}0} > 1$

# FLRW curvature constant

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  - azimuthal equidistant coords: $R_{\mathrm{C}}$

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- $\Omega_{\mathrm{tot}0} > 1$ *spherical*

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- $\Omega_{tot0} > 1$ *spherical* $R_C$ real

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- $\Omega_{tot0} > 1$ *spherical* $R_C$ real

- $\Omega_{tot0} = 1$ *flat*

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- $\Omega_\mathrm{tot0} > 1$   *spherical*   $R_\mathrm{C}$ real

- $\Omega_\mathrm{tot0} = 1$   *flat*   $R_\mathrm{C}$ undefined

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- $\Omega_{\mathrm{tot}0} > 1$   *spherical*   $R_{\mathrm{C}}$ real

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- $\Omega_{\mathrm{tot}0} < 1$

# FLRW curvature constant

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- $\Omega_{\mathrm{tot}0} > 1$  *spherical*  $R_C$ real

- $\Omega_{\mathrm{tot}0} = 1$  *flat*  $R_C$ undefined

- $\Omega_{\mathrm{tot}0} < 1$  *hyperbolic*

# FLRW curvature constant

■ metric in

◆ <u>azimuthal equidistant</u> coords: $R_{\mathrm{C}}$

◆ <u>orthographic</u> coords: $k$

■ orthographic: $1 - kr_\perp^2 = 0$ coord singularity at equator

■ $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$

■ <u>Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$</u> $\Rightarrow$ $\boxed{R_{\mathrm{C}} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}}$

■ $\Omega_{\mathrm{tot}0} > 1$ *spherical* $R_{\mathrm{C}}$ real

■ $\Omega_{\mathrm{tot}0} = 1$ *flat* $R_{\mathrm{C}}$ undefined

■ $\Omega_{\mathrm{tot}0} < 1$ *hyperbolic* $R_{\mathrm{C}}$ imaginary (or use $|R_{\mathrm{C}}|$)

# Einstein's free parameter: $\Lambda$

- Einstein: prevent expansion/contraction via $\Lambda$
  ADS:1917SPAW.......142E

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- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} + \mathbf{g}\Lambda = 8\pi\mathbf{T}$ and simplify:
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- ■ *hint:* mixed index form of $\mathbf{g}$ is easy

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- use $r =$ comoving radial distance

- consider $0 < z \ll 1$ limit $\Leftrightarrow 1 > a \gg 0.5$

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- high-level frontends (e.g. python) should be easy to write

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want to define $d_L$ such that flux is $F = \dfrac{L}{4\pi d_L^2}$

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- surface area of 2-sphere $= 4\pi r_\perp^2$ (comoving)

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- ■   <span style="color:green">w:Distance measures (cosmology)</span>

# FLRW distances e.g. $\Lambda$CDM



Comparison of Distance Measures 0 < z < 0.5

| | |
|---|---|
| top | Luminosity |
| | naive Hubble |
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| bottom | Angular diameter |

distance in Gly or age in Gyr

redshift z

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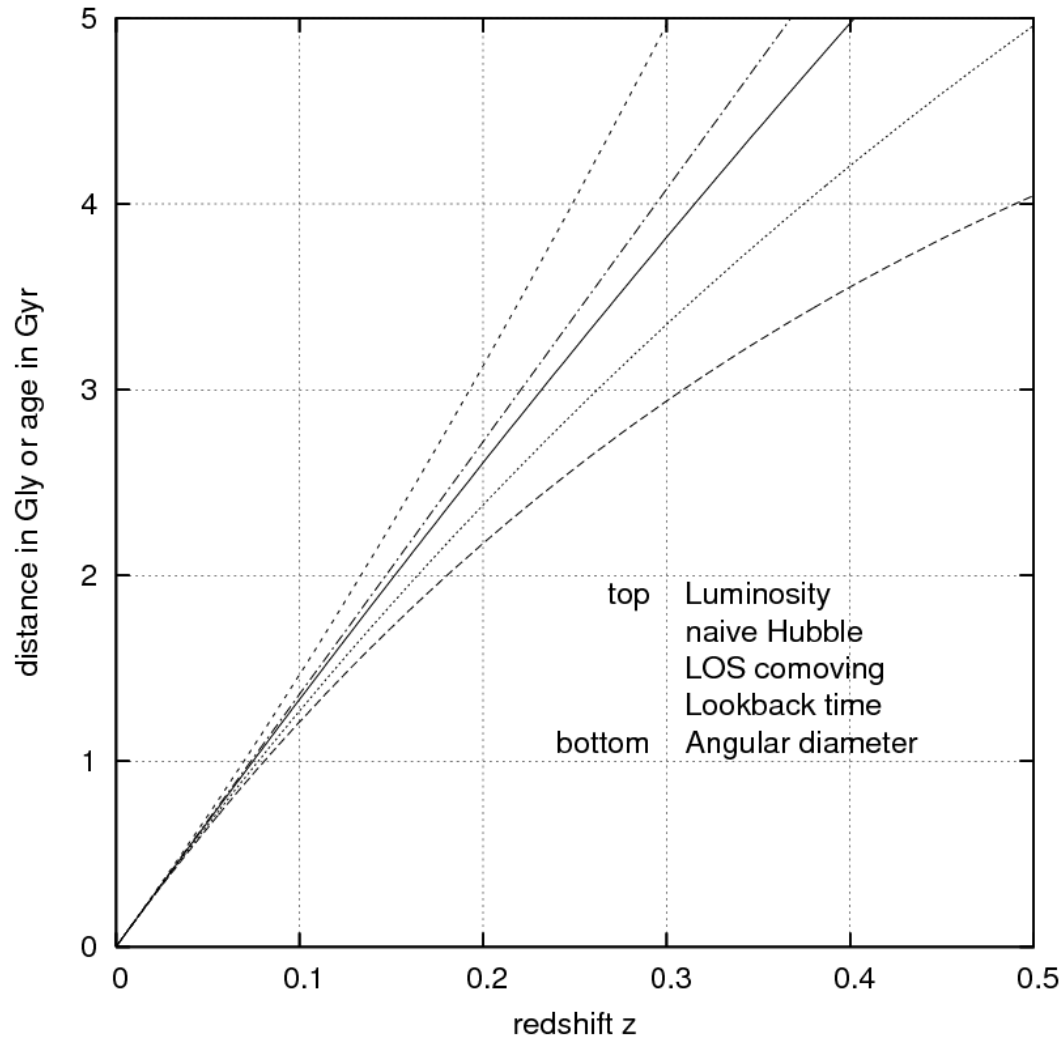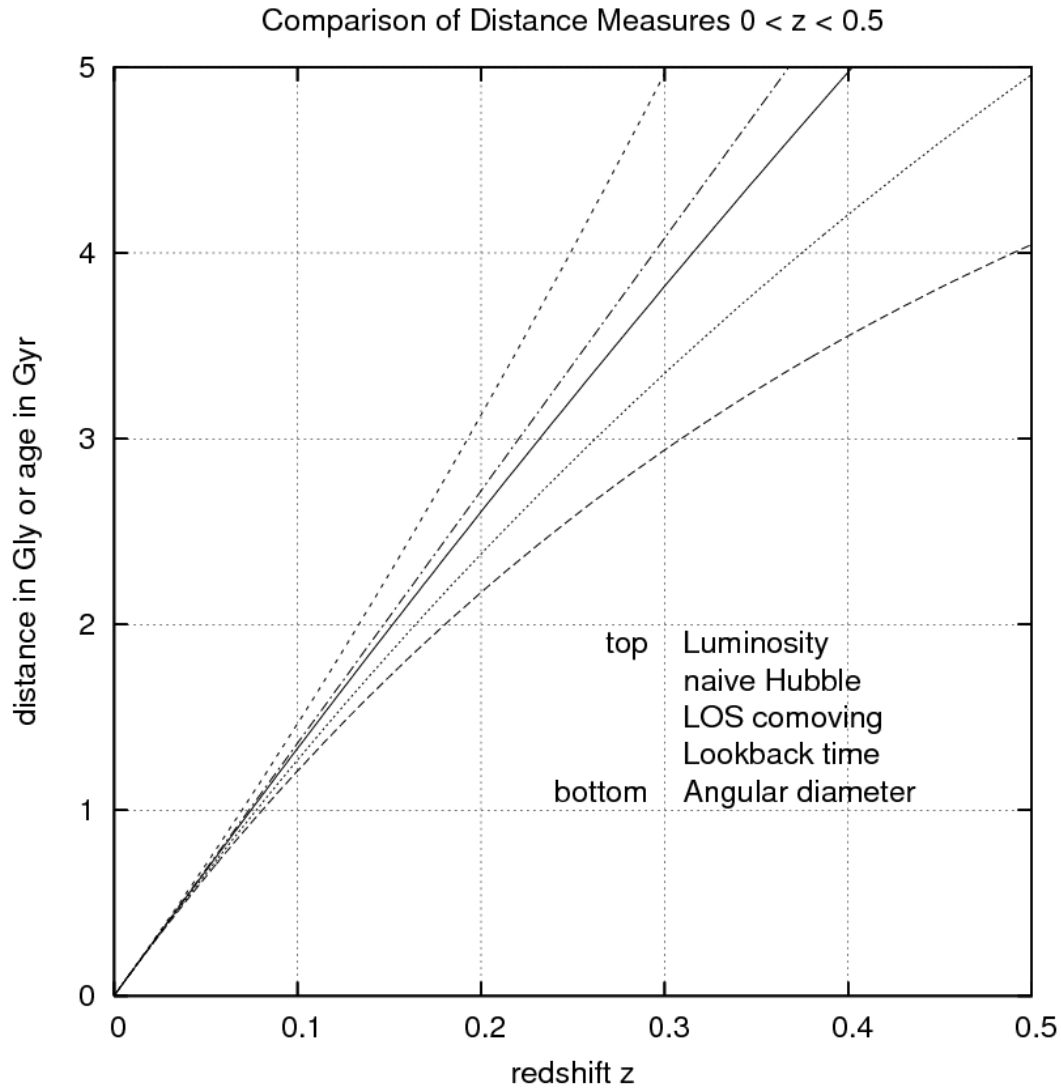
Defn: $\boxed{h := H_0/100 \text{ km/s/Mpc}}$ (without a "0" subscript on $h$)

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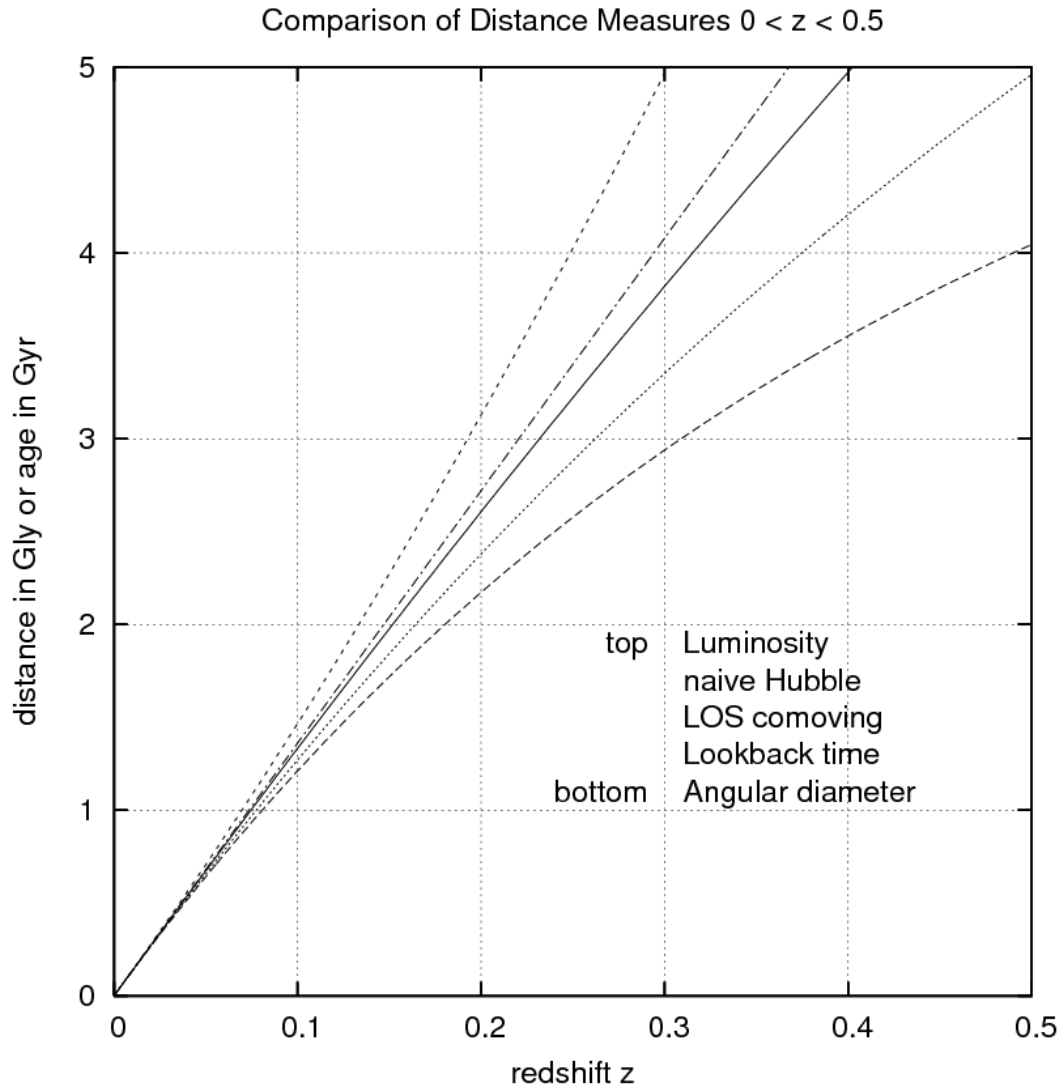
$$\Omega_{\mathrm{m}0} = 0.3, \ \Omega_{\mathrm{r}0} = 10^{-4}, \ \Omega_{\Lambda 0} = 1.0 - (\Omega_{\mathrm{m}0} + \Omega_{\mathrm{r}0}), \ h = 0.7, \ \Omega_{k0} = 0$$

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Comparison of Distance Measures 0 < z < 10,000
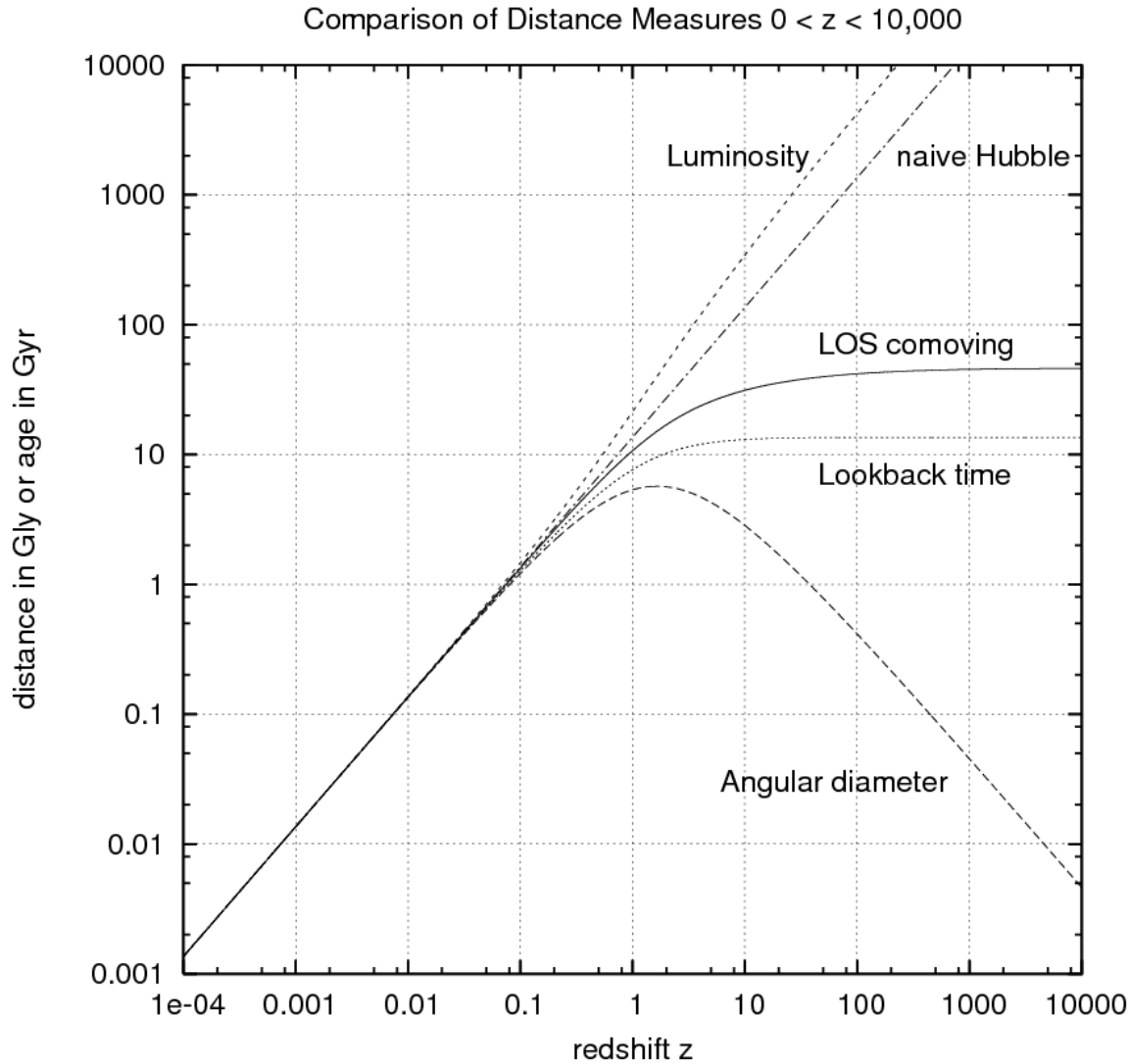
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$\Rightarrow$ no conflict with locally Lorentzian (SR) spacetime

# Non-radial spatial geodesics

■ What is the comoving distance between two objects at different celestial positions and different redshifts, for an arbitrary curvature $(+, 0, -)$?

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- distances on the 2-sphere, embedded in $\mathbb{R}^3$

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- $\boxed{\text{a distance in } S^2 = \text{arc-length in } \mathbb{R}^3:}$

$$
\boxed{r_{12} = R_{\mathrm{C}} \, \theta_{12} = R_{\mathrm{C}} \, \cos^{-1} \left[ \langle \mathbf{a}_1, \mathbf{a}_2 \rangle / R_{\mathrm{C}}^2 \right]}
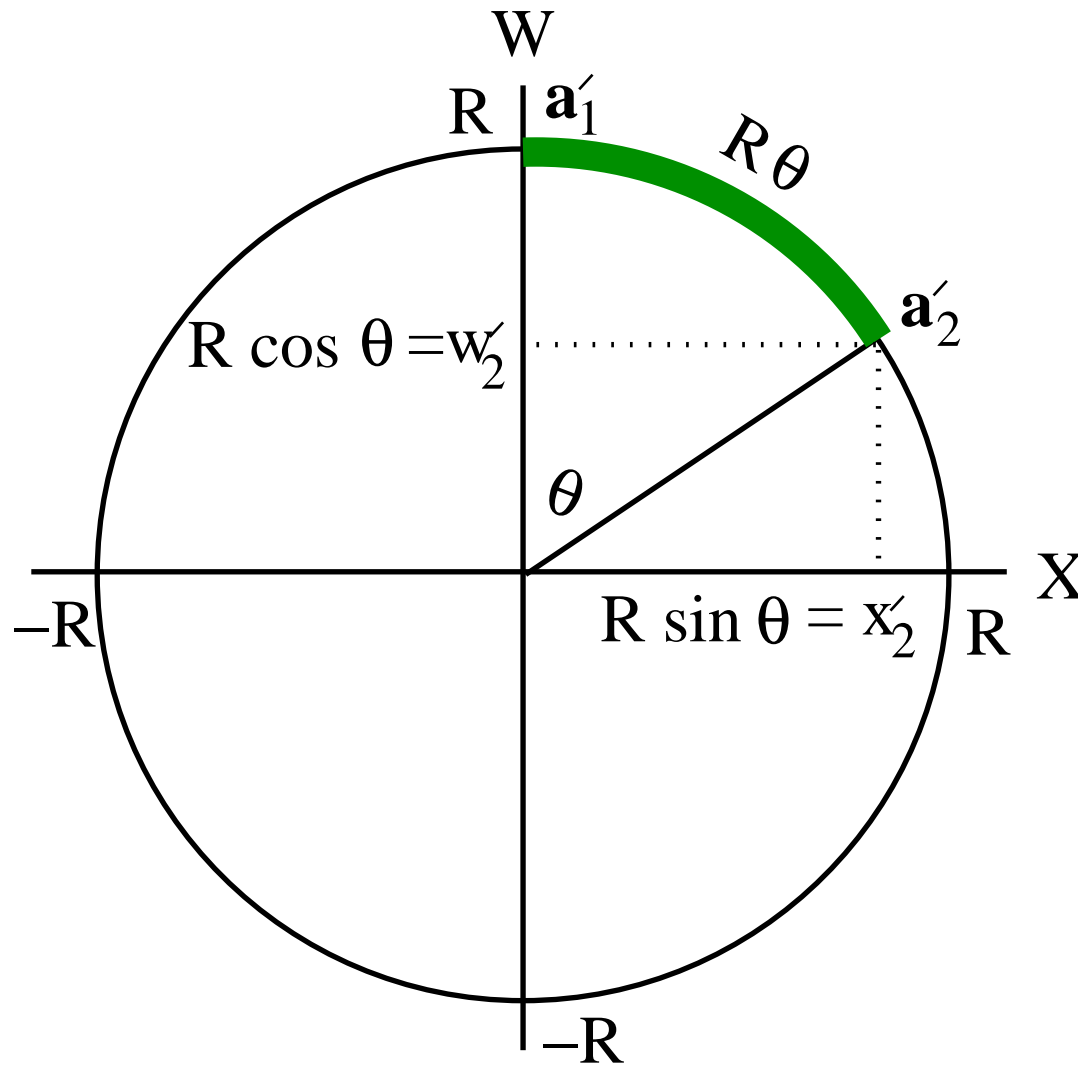$$

Warning: $r$ is a geodesic distance **on** $S^2$; it is **not** a radial distance from the $S^2$ "centre".

# Non-radial spatial geodesics

- positive curvature

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■ negative curvature

# Non-radial spatial geodesics

$R \cosh \phi = w_2'$

$\mathbf{a}_2'$

$\mathbf{a}_1'$

W

R

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X

# distances on $\mathsf{S}^3 \subset \mathbb{R}^4$ or $\mathsf{H}^3 \subset \mathsf{M}^4$

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■ metric on $\mathsf{S}^3$ (or $\mathbb{R}^3$ or $\mathsf{H}^3$):

$$\mathrm{d}s^2 = \begin{cases} (k/|k|) \ (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) + \mathrm{d}w^2 & k & \neq & 0 \\ \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 & k & = & 0 \end{cases}$$

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■ inner product:

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  ◆ Fourier mode power spectrum $P(k)$

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homogeneity