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A Solution to the Isolatitude, Equi-area, Hierarchical Pixel-Coordinate System

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Abstract. Górski et al. (1999b) have earlier presented the outline of a pixelisation-to-spherical-coordinate transformation scheme which simultaneously satisfies three properties which are especially useful for rapid analyses of maps on a sphere: (i) equal spacing of pixels along lines of constant latitude, (ii) equal pixel 'areas' (solid angles) and (iii) hierarchical scaling with increasing numbers of pixels. Their outline is based on the division of the sphere into twelve regions covering equal solid angles, which are hierarchically subdivided in a way compatible with these three criteria. In this paper, a complete derivation of this scheme is presented, including, in particular, (1) the angle θ^* defining the limit between polar and equatorial regions, and (2) the transformations from the unit interval $[0,1] \wedge [0,1]$ to spherical coordinates in a polar region.

Key words. cosmic microwave background – Cosmology: observations – Methods: data analysis – Methods: observational

1. Introduction

The most commonly required analyses of microwave background data require spherical harmonic analyses of temperature fluctuation maps. Given increasingly large numbers of pixels, Górski, Hivon, & Wandelt (1999a); Górski et al. (1999b) pointed out that an ideal pixelisation-to-spherical-coordinate transformation scheme would satisfy the following requirements:

(i) an isolatitude requirement, i.e. pixels should be equally spaced in latitude,
(ii) pixel solid angles (hereafter, 'areas') should be equal, and
(iii) the system should scale hierarchically with increasing numbers of pixels, preferably according to some power of two, in order to maximally exploit the binary nature of present computer technology.

Górski et al. (1999b) present the outline of a system satisfying these requirements. The idea of dividing the sphere into 12 regions — four equatorial regions and eight polar regions — which are mapped to or from the unit interval $[0,1] \wedge [0,1]$, is presented there.

However, the full solution is not derived in Górski et al. (1999b).

In this paper, firstly, in Sect. 2.1, the angle θ^* defining the limit between polar and equatorial regions of this scheme is derived.

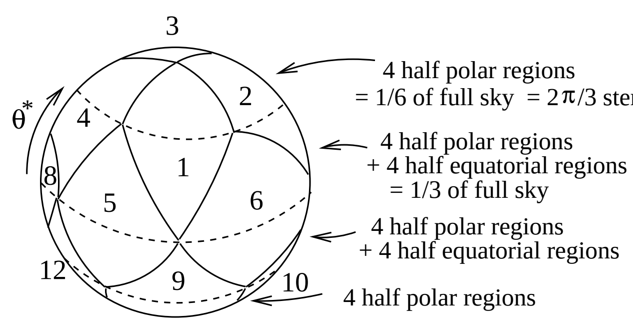

Fig. 1. Schematic diagram of the 12 regions, or equivalently, 24 half regions, showing how the isolatitude requirement and the equal pixel area requirement together require that the solid angle from the line of constant latitude θ^* to the pole is $2\pi/3$ ster.

As is described in Górski et al. (1999b), the binary bits of fractions with the square unit interval $[0,1] \wedge [0,1]$ are used for pixel ordering within a region. For completeness, we remind the reader of this ordering in Sect. 2.2 and express it algebraically.

In order to have the complete transformation from pixel numbers to spherical angles, the transformation between the fractional position (f_i, f_j) in the square unit interval and a pair of spherical angles (ϕ, θ) is needed. In Sect. 2.3.1, this transformation is derived for equatorial regions, and in Sect. 2.3.2, the transformation is derived for polar regions.

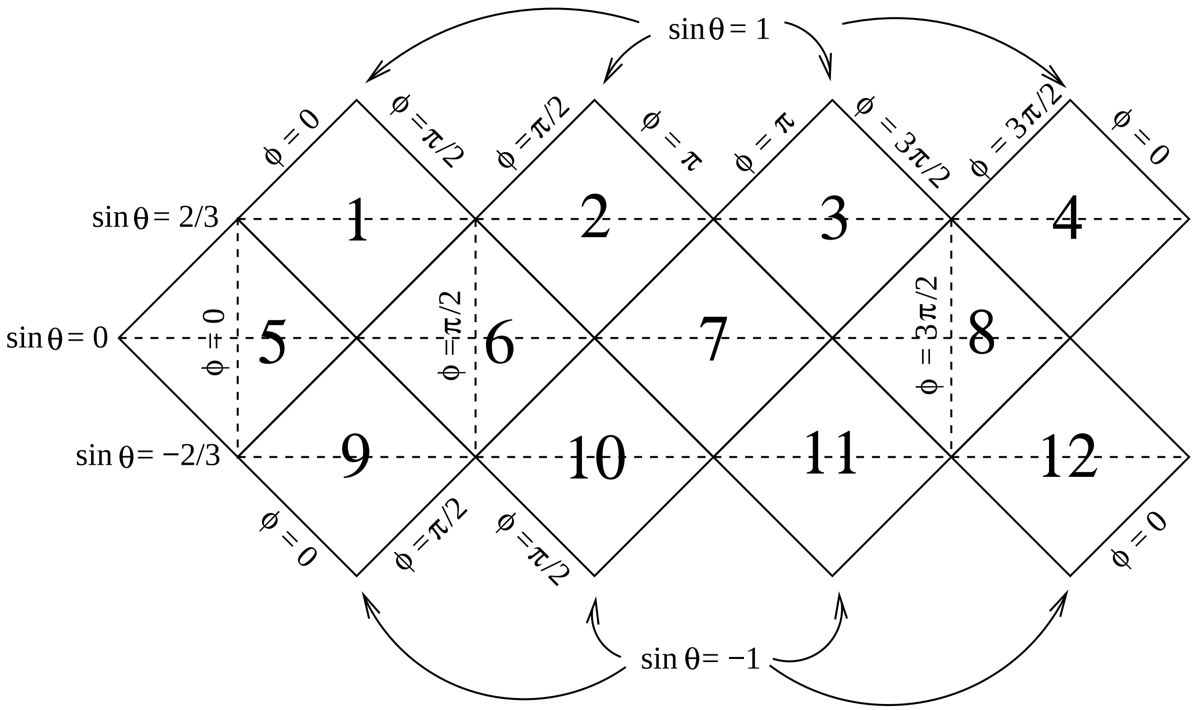

Fig. 2. Ordering of the 12 regions, shown in a projection where the full sky, 4π ster, is shown as 12 square unit intervals. Each pole is represented as a set of four distinct points in this projection, indicated by arrows. The derivation of the straight line (in this projection) dividing the "polar" regions into "polar" and "equatorial" parts, labelled $\sin\theta = 2/3$, is derived in Eq. (7).

The convention chosen here for spherical coordinates is to have a longitude coordinate

$$\phi \in [0, 2\pi] \tag{1}$$

and a latitude coordinate

$$\theta \in [-\pi/2, \pi/2]. \tag{2}$$

For clarity of presentation, it is also useful to define

$$\phi_\pi \equiv \phi/\pi \tag{3}$$

and

$$s \equiv \sin\theta. \tag{4}$$

2. Calculations

2.1. The boundary between equatorial and polar regions: θ^*

Fig. 1 (cf. fig. 2 of Górski et al. (1999b)) and Fig. 2 show the overall geometry of the 12 regions, including an ordering, though any other ordering could easily be chosen.

In order for the 12 regions to have equal pixel areas, the coordinates of the four vertices along the equator, of the equatorial regions, must be spaced by an interval of $\Delta\phi = \pi/2$, i.e. they are

$$\phi_\pi = 1/4,\ 3/4,\ 5/4,\ 7/4, \theta = 0. \tag{5}$$

The two polar vertices are

$$\phi_\pi \text{ undefined},\ \theta = -\pi/2, +\pi/2. \tag{6}$$

Finding the coordinates of the remaining eight vertices, which separate equatorial and polar regions, requires an integration.

Define θ^* to be the angle from the equator to the curve of constant latitude separating the equatorial regions from the "polar" half of the polar regions, as shown in Fig. 1. (The "polar" regions are, in fact, divided between a "polar" half which is polar and an "equatorial" half which uses a transformation similar to that of the "equatorial" regions. This is explained below in Sect. 2.3.2.)

Since the system is an isolatitude, equal pixel area system, the entire solid angle from a curve of constant latitude θ^* to the pole must be half of the total solid angle represented by four regions, i.e. half of one third of the full sky, i.e. one sixth of 4π steradians, i.e. $2\pi/3$ ster. This is illustrated in Fig. 1.

Hence,

$$\int_0^{2\pi} \mathrm{d}\phi \int_{\theta^*}^{\pi/2} \cos\theta \,\mathrm{d}\theta \ = \ 2\pi/3,$$

$$\Rightarrow 2\pi \left[\sin\theta\right]_{\theta^*}^{\pi/2} \ = \ 2\pi/3$$

$$\Rightarrow 2\pi(1 - \sin\theta^*) \ = \ 2\pi/3$$

$$\Rightarrow \sin\theta^* \ = \ 2/3 \tag{7}$$

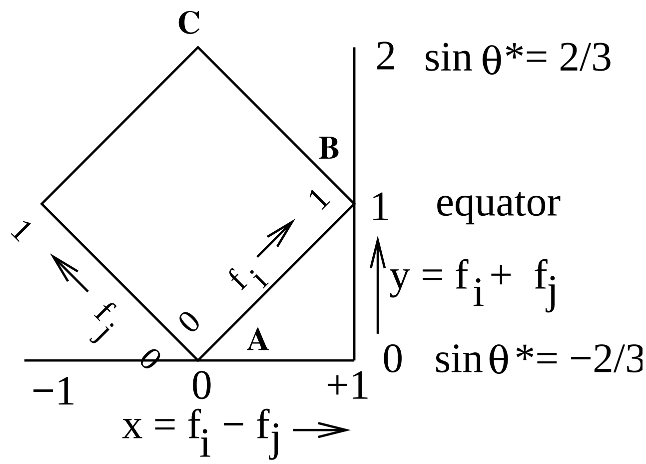

Fig. 3. Equatorial region: unit interval $[0,1] \wedge [0,1]$ parametrised by fractions f_i, f_j and linearly transformed to $x \equiv f_i - f_j$ and $y \equiv f_i + f_j$. This can be any of the regions 5, 6, 7, 8; without loss of generality, the discussion in Sect. 2.3.1 is presented for region 5.

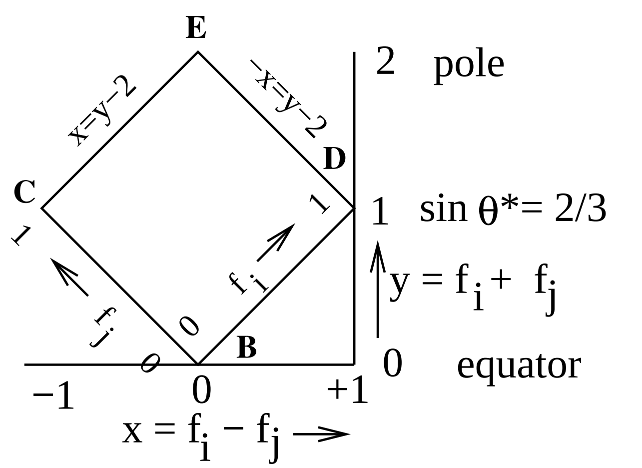

Fig. 4. Polar region: unit interval $[0,1] \wedge [0,1]$ parametrised by fractions f_i, f_j and linearly transformed to $x \equiv f_i - f_j$ and $y \equiv f_i + f_j$. This can be any of the regions 1, 2, 3, 4, 9, 10, 11, 12; without loss of generality, the discussion in Sect. 2.3.2 is presented for region 1.

$$\Rightarrow \theta^* = \sin^{-1} 2/3 \approx 41.8°. \tag{8}$$

Hence, in Fig. 2, the curve of constant latitude dividing each polar region into two equal halves is labelled $\sin\theta = \pm 2/3$.

Note that θ^* is *not* $\pi/4$, even though a casual glance at Fig. 1 and casual intuition might suggest this.

Hence, the eight vertices separating equatorial and polar regions are

$$(\phi_\pi = 0,\ 1/2,\ 1,\ 3/2,\ \theta = \theta^*),$$
$$(\phi_\pi = 0,\ 1/2,\ 1,\ 3/2,\ \theta = -\theta^*). \tag{9}$$

2.2. Hierarchical ordering of pixels

Requirement (iii), the hierarchical ordering, is a binary ordering within the (square) unit interval, and an ordering of the twelve unit intervals. This ordering is referred to as 'nested' by Górski et al. (1999b), and is the ordering used in the data files of the first year WMAP data (Bennett et al. 2003), where ϕ and θ are galactic longitude and latitude respectively. It is shown for two different resolution levels in the lower half of fig. 3 of Górski et al. (1999b).

2.2.1. Binary ordering of pixels within the square unit interval

Figs 3, 4 show the definitions of f_i, f_j as fractions in the unit interval in an equatorial and a polar region respectively.

Let us write f_i, f_j as binary fractions, rounded to the nearest pixel centre, to some precision $n \geq 1$, corresponding to linear pixel sizes of *approximately*, but not exactly, $\frac{\pi/2 \text{ rad}}{2^n}$, or pixel *areas* of *exactly* $\frac{\pi/3 \text{ ster}}{4^n}$:

$$0 \leq f_i = 0.b_{2n-1}...b_3b_11, f_j = 0.b_{2n}...b_4b_21 \leq 1 \tag{10}$$

where $b_i \in \{0,1\}$. The last binary place is occupied by "1" in order to have a pixel centre. Then the bits in f_i and f_j provide *alternating bits* for integer ordering of pixels within the square unit interval.

This mapping is sufficient for both equatorial and polar regions.

This is shown in fig. 1 of Górski et al. (1999b).

For example, with just one binary digit, a region is divided into four pixels. If we have binary representations

$$f_i = 0.b_11, f_j = 0.b_21, \tag{11}$$

then the pixel centred at (f_i, f_j) is the $(b_2b_1)_2$-th pixel within the region.

With two binary digits, a region is divided into sixteen pixels. If we have binary representations

$$f_i = 0.b_3b_11, f_j = 0.b_4b_21, \tag{12}$$

then the pixel centred at (f_i, f_j) is the $(b_4b_3b_2b_1)_2$-th pixel within the region.

More generally, with $n \geq 1$ binary digits, the pixel at (f_i, f_j)

$$f_i = 0.b_{2n-1}...b_3b_11, f_j = 0.b_{2n}...b_4b_21, \tag{13}$$

is the $(b_{2n}...b_2b_1)_2$-th pixel (again in binary) in the region.

2.2.2. Ordering of the twelve square unit intervals

The ordering of the twelve regions shown in Fig. 2 is that used in the first year WMAP data files. This combines with the ordering within a region.

That is, with $n \geq 1$ binary digits, the pixel centred at (f_i, f_j) in the k-th region, with binary representations

$$f_i = 0.b_{2n-1}...b_3b_11, f_j = 0.b_{2n}...b_4b_21, \tag{14}$$

Table 1. Vertex conversions for an equatorial region. Equations (16), (3) and (4) define x, y, ϕ_π and s. See eqns (5), (9) and (7) for the values of ϕ_π and s.

label	x	y	ϕ_π	s
A	0	0	0	$-2/3$
B	1	1	1/4	0
C	0	2	0	2/3

(17)

is pixel number

$$\left[(k-1)\ 2^{2n}\right]_{10} + \left[b_{2n}...b_2 b_1\right]_2 \tag{15}$$

where the first term is shown in decimal notation and the second in binary notation.

2.3. Transformations from the unit interval to ϕ, θ

For either an equatorial or polar region, shown in Figs 3, 4 respectively, it is useful to make the change of variables

$$\begin{aligned} x &\equiv f_i - f_j \\ y &\equiv f_i + f_j, \end{aligned} \tag{16}$$

where f_i, f_j are the fractions in the unit interval.

The requirements of

(i) isolatitude pixel positions, and
(ii) equal pixel areas

then require transformations from (f_i, f_j) to (ϕ, θ), or equivalently from (x, y) to (ϕ, θ), or even more simply from (x, y) to (ϕ_π, s), as defined in eqs (3), (4).

Use of equal intervals in $s \equiv \sin\theta$ in latitude enables straightforward linear mappings of the complete equatorial band (the regions labelled here as "equatorial", plus half of each of the regions labelled here as "polar"). This can be thought of as stretching the latitude coordinate in order to compensate for shorter lengths of curves of constant latitude.

However, this cannot function close to the poles, hence, a distinct mapping is required for the polar regions.

2.3.1. Equatorial regions: transformations from unit interval to ϕ, θ

Since

$$\mathrm{d}(\sin\theta) = \cos\theta\ \mathrm{d}\theta, \tag{18}$$

the mapping in any of the four equatorial regions is provided by a linear map from x to ϕ and from y to $\sin\theta$.

Without loss of generality (there are four equatorial regions in total), it is sufficient to consider the region labelled "5" in Figs 1, 2 to be the region which has $\phi = 0$ at its centre.

Then, the definitions (16), (3) and (4) together with the values in eqns (5), (9) and (7) give the vertex conversions shown in Table 1.

The requirements of

(i) isolatitude pixel positions, and
(ii) equal pixel areas

together with the vertex conversions in Table 1 yield the linear transformations

$$\begin{aligned} \phi_\pi &= x/4 \\ s &= \frac{2}{3}(y-1). \end{aligned} \tag{19}$$

Table 2. Vertex conversions (B to E) and a boundary condition (F) for a polar region. Equations (16), (3) and (4) define x, y, ϕ_π and s. See eqns (6), (9) and (7) for the values of ϕ_π and s.

label	x	y	ϕ_π	s
B	0	0	1/4	0
C	-1	1	0	2/3
D	$+1$	1	1/2	2/3
E	0	2	$0 \le \lim_{(x,y)\to \mathrm{E}} \phi_\pi \le 1/2$	1
F	$\phi_\pi(-x = y-2) = \phi_\pi(x = y-2) + 1/2$			

(20)

2.3.2. Polar regions: transformations from unit interval to ϕ, θ

Without loss of generality, the polar region "1", as shown in Figs 1, 2, adjacent to the equatorial region labelled "5", is considered.

The vertex conversions for this polar region are those in Table 2.

Vertex conversion Eq. (20).F is a requirement of continuity of ϕ_π between adjacent polar regions: the two boundaries of the upper half of the region are the lines $x = y-2$ and $-x = y-2$ (respectively upper left and upper right boundaries of the unit interval in Fig. 4) and for any fixed value of y, must cover the angular interval $[0, \pi/2]$, i.e. the ϕ_π value at constant y must increase by $\Delta\phi_\pi = 1/2$ between these two limits. Hence, Eq. (20).F.

The region is illustrated in Fig. 4.

The lower half of the 'polar' region, where $y \le 1$, is really in the equatorial part of the sphere, so a similar linear solution applies as above:

$$\begin{aligned} \phi_\pi &= \frac{1+x}{4} \\ s &= \frac{2y}{3}, \end{aligned} \tag{21}$$

providing continuity and satisfying the isolatitude and equi-pixel requirements and the vertex conversions (20).B, (20).C and (20).D in Table 2.

By continuity, (20).C and (20).D must also be satisfied for limiting cases in the upper half of the region as $y \to 1^+$.

The isolatitude and equi-pixel requirements imply that for $y \ge 1$, ϕ_π is linear in x, and in order to satisfy (20).C and (20).D, it follows that

$$\phi_\pi = \frac{1 + f(y)x}{4} \tag{22}$$

for some function f which does not depend on x and which satisfies

$$f(y=1)=1 \tag{23}$$

in order to be continuous with the region $y \leq 1$.

Boundary condition Eq. (20).F (Table 2) is

$$\phi_\pi(-x=y-2)=\phi_\pi(x=y-2)+1/2 \tag{24}$$

$$\Rightarrow \frac{1+f(y)[-(y-2)]}{4}=\frac{1+f(y)(y-2)}{4}+1/2 \tag{25}$$

$$\Rightarrow f(y)[-(y-2)]=f(y)(y-2)+2 \tag{26}$$

$$\Rightarrow f(y)=\frac{1}{2-y}. \tag{27}$$

Because boundary condition F implicitly includes conditions C and D when $y=1$, the required constraint in Eq. (23) is satisfied, i.e. $f(1)=1$.

Thus,

$$\phi_\pi=\frac{2-y+x}{4(2-y)}. \tag{28}$$

To obtain s, note that equality of pixel areas for $y \geq 1$ and $y \leq 1$ requires that

$$\left.\frac{\partial s}{\partial y}\right|_{y\geq 1}\left.\frac{\partial \phi_\pi}{\partial x}\right|_{y\geq 1}=\left.\frac{\partial s}{\partial y}\right|_{y\leq 1}\left.\frac{\partial \phi_\pi}{\partial x}\right|_{y\leq 1}. \tag{29}$$

Equations (28), (21) then give

$$\left.\frac{\partial s}{\partial y}\right|_{y\geq 1}\frac{1}{4(2-y)}=\frac{2}{3}\,\frac{1}{4} \tag{30}$$

$$\Rightarrow \left.\frac{\partial s}{\partial y}\right|_{y\geq 1}=\frac{2}{3}(2-y). \tag{31}$$

The isolatitude condition implies that

$$\frac{\partial s}{\partial x}=0, \tag{32}$$

so integration gives

$$s=\text{constant}-\frac{1}{3}(2-y)^2, \tag{33}$$

and vertex conversion D provides the constant, so that

$$s=1-\frac{1}{3}(2-y)^2. \tag{34}$$

Taking the limit of ϕ_π as $(x,y)\rightarrow(0,2)$, within the range $y-2 \leq x \leq 2-y$, clearly provides vertex conversion E.

3. Conclusion

The solution to the regionalisation of the sphere recommended by Górski et al. (1999b), with the three requirements

(i) isolatitude, i.e. equal spacing of pixels in latitude, along lines of constant latitude,
(ii) equal pixel areas, and
(iii) hierarchical ordering of pixels within square unit intervals which map to equal solid angles,

has been completed by deriving, in particular, the angle θ^* separating polar and equatorial regions, and the full transformations from the unit interval $[0,1]\wedge[0,1]$ to spherical coordinates in a polar region.

Together with the binary ordering presented by Górski et al. (1999b), the full solution is as follows.

Binary, bitwise mappings relate fractional positions (f_i, f_j) within the k-th square unit interval to a pixel number. With $n \geq 1$ binary digits, the pixel centred at (f_i, f_j) with binary representations

$$f_i=0.b_{2n-1}...b_3b_1 1,\, f_j=0.b_{2n}...b_4b_2 1, \tag{35}$$

where $b_i \in \{0,1\}$, in the k-th region shown in Fig. 2, is pixel number

$$\left[(k-1)\;2^{2n}\right]_{10}+\left[b_{2n}...b_2b_1\right]_2, \tag{36}$$

where the first term is shown in decimal notation and the second in binary notation, as mentioned above in Eq. (15).

The vertices along the equator of the equatorial regions are (from Eq. (5))

$$\phi_\pi=1/4,\ 3/4,\ 5/4,\ 7/4, \theta=0; \tag{37}$$

the vertices separating equatorial and polar regions are (from Eq. (9))

$$\begin{gathered}(\phi_\pi=0,\ 1/2,\ 1,\ 3/2,\ \theta=\theta^*),\\(\phi_\pi=0,\ 1/2,\ 1,\ 3/2,\ \theta=-\theta^*);\end{gathered} \tag{38}$$

and the two polar vertices are (from Eq. (6))

$$\phi_\pi \text{ undefined},\ \theta=-\pi/2,+\pi/2. \tag{39}$$

Mappings from $(x\equiv f_i-f_j, y\equiv f_i+f_j)$ to $(\phi_\pi \equiv \phi/\pi, s\equiv \sin\theta)$, are found, without loss of generality, for one equatorial region and one polar region:

$$\left.\begin{array}{rcl}\phi_\pi&=&\frac{x}{4}\\ s&=&\frac{2}{3}(y-1)\end{array}\right\}\ \text{equatorial region}$$
$$\left.\begin{array}{rcl}\phi_\pi&=&\frac{1+x}{4}\\ s&=&\frac{2y}{3}\end{array}\right\}\ \text{polar region, } y\leq 1$$
$$\left.\begin{array}{rcl}\phi_\pi&=&\frac{2-y+x}{4(2-y)}\\ s&=&1-\frac{1}{3}(2-y)^2\end{array}\right\}\ \text{polar region, } y\geq 1 \tag{40}$$

summarised here for convenience from equations (19), (21), (28) and (34).

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